1. MATHS ANXIETY

§1.1. Problems With Mathematics

OK, you thought you could get away with never having to open another maths textbook, and here you are. You're passionate about following some particular line of study and you've been told you now need some maths. The last time you did any maths was probably some years ago and you're anxious about having to do maths again. Probably you never did understand it properly when you dropped mathematics, and most of what little you did understand you've forgotten.

If this sounds patronising, then probably you've no need to be reading these notes. They're not designed for the student who just needs to brush up their knowledge, but rather for those who are afraid they might never be able to conquer their fear of maths.

At least you're several steps ahead of the junior high school student who, not only doesn't understand maths, but really can't be bothered. That might have been you once! But now you're highly motivated. Like it or not you *need* maths to do advanced chemistry, or environmental science, or forensic science or something really interesting.

Maths anxiety is a very common syndrome. Usually it can be cured by a careful explanation of the basic concepts. In rare cases it's a serious disorder and needs the help of a psychologist, or even a psychoanalyst. But in those cases there are usually other symptoms. Phobias often come in multiple forms.

So I'm assuming that the fact that you're reading on means that you're a normally well-adjusted adult who's just scared of maths. Coming back to it after some years gives you two advantages. You're more motivated and you're more mature. You might still be mathematically immature but no doubt you're mature intellectually in other areas.

So why does mathematics have such a bad reputation? As a mathematician, I find that at social occasions when it comes out that I teach mathematics I nearly always get some outburst about how the other person hated maths at school, or was never very good at it. If I hear my friend say they're a professor of psychology, or astronomy, or literature, by contrast the response is usually "how interesting".

Many people almost boast that they "can't do maths", whereas this is not the case with speaking German, understanding political history or playing cricket.

The problem with maths, compared to practically every other area of study, is that it's linear. It builds on previous knowledge to an enormous extent. If you miss the first ten minutes of a history lecture you can usually pick up the thread. Miss the first ten minutes of a maths lesson and the chances are you will be perplexed for the rest of the hour. Mathematics, as an area of study, is quite unique.

§1.2. Mathematics is an Ancient Discipline

Somebody once said that mathematics is the second oldest profession! Certainly its roots go back much further than any of the sciences. Physics and chemistry go back only a few hundred years. Psychology is much younger. Mathematics began with prehistory. Moreover nothing in mathematics has been superseded by later theories. Discoveries have been built upon previous ones so that the body of knowledge known as mathematics is the most mature of all areas of knowledge.

Probably almost all of what you know in mathematics was known in the middle ages. If you had done the highest level of school mathematics it would only have brought you up to the time of Elizabeth I. Doing a bachelor's degree in mathematics might, with a few exceptions, bring you up to the time of Queen Victoria. When people hear about research in mathematics they often reply, "wasn't it all worked out a long time ago?" It comes as a surprise that new mathematics is being discovered at an ever increasing rate. The *Mathematical Reviews*, is a journal that lists short summaries, or abstracts, of the more important mathematical papers. It's now on line, but the last time paper copies were published many years ago, each month's issue was as big as a telephone book. Whole new branches of mathematics have sprung up in the last sixty years.

Once you realise that there's so much to know it's a bit intimidating. Fortunately you'll only need to know a tiny fraction of what's out there. And, although everything you'll learn in these notes was known six hundred years ago, at least we now have become more efficient in teaching it. Indeed six hundred years ago you would have learnt this material at university, whereas these days it's taught in junior high school.

§1.3. Mathematics as Storytelling

"What I like about mathematics", somebody once told me, "is that everything is black and white. You know where you stand. Mathematical truth is absolute." They were no doubt thinking of disciplines such as history, or psychology where there are conflicting schools of thought, and even facts are disputed. Ask a physicist whether light consists of particles or waves and he or she will say, "well, it depends on how you look at it." Once, atoms were believed to be indivisible particles like billiard balls. Now physicists haven't got to the bottom of the sub-atomic particles from which they're formed.

Yet, in a sense, mathematics has no facts. Everything is relative. It says that if you make certain assumptions then such and such must logically follow. But there are no absolute truths in mathematics!

But surely “1 + 1 = 2” is an absolute fact! Certainly mathematicians long ago developed a system of numbers where 1 + 1 = 2, and this system seems to be useful in the real world when we want to count things. But mathematicians have also invented a system where 1 + 1 = 0, and this has proved extremely useful in computer science.

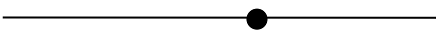
Mathematicians are the storytellers of the scientific world. They create great myths. Now myths can convey truth, even if the events are not true in the strict historical sense. Myths are about worlds other than our own material world and these worlds live inside the human imagination.

The same is true of mathematics. It lives only in the imagination. After all, mathematicians happily talk of a perfectly round circle or an infinitely long line. Neither exists in the real world. We might draw a pretty good circle on a piece of paper with geometrical instruments, but examine it under a microscope and you’ll see that it’s far from perfectly round. Moreover it will have thickness, which a mathematical circle does not. You can represent an infinitely long line on paper, but the dimensions of the page will limit how long it actually is.

Mathematicians create imaginary worlds. It’s an intrinsically abstract intellectual pursuit. Somebody once said that if the universe suddenly disappeared the only areas of knowledge that would survive are mathematics and theology. Geology needs rocks, astronomy needs stars. Mathematics is independent from the material world.

Of course, it could be argued that if mathematics exists in the human mind it will disappear once the human race dies out. But then there are those who claim that God invented mathematics and He was there before creation. I'm not taking sides on this question! You signed up to learn some mathematics, not theology or philosophy. The point I'm making is that mathematics makes up stories which scientists are free to use if they find it helps.

Take the question of the angles of a triangle. They always add up to 180 degrees. Well, Euclid stated some axioms from which Euclidean geometry can be developed, and in that geometry it can be proved mathematically that the angles of a triangle always total 180 degrees. These axioms were considered to be self-evident. But in the 1800s mathematicians questioned one of these axioms. The parallel postulate states that given any line, and a point that's not on that line, there is exactly one line that passes through the given point and is parallel to the given line.



They wondered what would happen if there was more than one line through the point parallel to the given line, or perhaps no such line in some cases. By modifying the parallel postulate they developed new geometries where the angles of a triangle can be shown to total more than 180 degrees, or less than 180 degrees.

But surely, using careful measurement, we can prove that you always get 180 degrees. We might do that, but in so doing we would have stopped being a mathematician and started acting like an experimental physicist. One problem with experimental science is that you are limited by the accuracy of your instruments. Perhaps the angles of a triangle always add up to 179.999999 degrees. Perhaps the discrepancy will only show up if the sides of the triangle can only be measured in light years. (A light year is not a unit of time but a very large unit of measurement – the distance that light travels in one year.)

The other problem with experimental science is that we can only test the hypothesis a finite number of times. What if we could measure angles exactly and in every case that we tried the angles added up to exactly 180 degrees? That doesn't prove that it works for all the triangles we haven't tried.

Mathematicians have developed many geometries in the world of their imagination. They say to scientists, here are some mathematical models for you to investigate. It's the role of the experimental scientist to decide which one seems to work best in the material world. And which one have they settled on? For most scientific purposes they're happy with the good old flat geometry of Euclid. But very close to an atom there is experimental evidence that space is "curved" and scientists have adopted one of the non-Euclidean geometries to describe this phenomenon.

§1.4. Left Brain or Right Brain?

The human brain is divided into two hemispheres. The left brain controls the right side of the body, and vice versa. But psychologists have a theory that people with a dominant left hemisphere are strong logically while those with a dominant right hemisphere are more creative and imaginative. For this reason many people say "I can't do maths because I'm more right brain."

It's true that mathematics requires logical thinking. But it also requires imagination. After all, the concepts of mathematics are all imaginary. Take the number '3' for example. You've never held the number 3. Although you've seen the symbol '3' you've never seen the number 3 itself. The symbol '3' is just a name for the abstract entity 3. After all, just because you've seen the word 'mermaid' doesn't mean you've seen an actual mermaid!

Mathematics is as much right brain as it is left brain. In some universities you can major in mathematics as part of an arts degree or as part of a science degree. Maths is *both* arts and science.

Yes, logical thinking is vital if you are going to study mathematics. But logical thinking is just the ability to follow rules. If you can find your way round a smart phone or play cards you have no trouble with logical thinking. Logical thinking just requires being disciplined.

If you're not prepared to follow the rules of a game, and constantly say, "well I should be allowed to trump that card" you'll have problems with mathematics. There are certain rules and you must follow them. Only when you get to be a research mathematician do you get to make up, or modify, the rules.

But being able to follow rules is not enough to give you the ability to do mathematics. I once thought that anybody could do mathematics provided they had a patient teacher who could explain the rules clearly. That was until my daughter did maths for the higher school certificate. I discovered that I could explain a question clearly, step by step, and she would understand it. But when I gave her a very similar question to do she was stumped. "I understood the previous question but you haven't shown me how to do *this* one." I would reply "can't you adapt what you learnt on that previous question to this one? The methods you need to use are essentially the same."

Now my daughter didn't lack creative thinking. It's just that she thought she had to leave creative thinking behind. Mathematics involves a combination of logical thinking and creative thinking.

It's often said that the reason why the modern generation is so bad at mathematics is because they're allowed to use calculators and they're not sufficiently drilled in their times tables. There's some truth in that ... but not much.

Let me make the bold suggestion that the drop in mathematical ability is because we no longer teach formal grammar! There's some truth in that, though probably not much. But what children used to get a lot of was the ability to recognise similes and metaphors.

Metaphor is what mathematics is built on. Perhaps not the poetic kind, as in "the clouds closed across the moon like gossamer curtains", but metaphor is intrinsic to mathematics. The ability to see when a certain question is "essentially the same as one you've just seen" is vital.

Recognising that two things are different in some respects but the same in others is what makes the metaphor. Of course the sun is not a warrior in a chariot. But as it moves across the sky we can think of a chariot racing across our field of view. We mightn't think of it spontaneously, but when we read it in a poem we should be able to respond by saying "I see what he means." The person with a purely logical mind would say "don't be silly. A warrior and his horse couldn't live out there in space – there's no air!"

When you look at the moon you probably think of its rocky landscape and its craters. But if someone mentions "the man in the moon" you should be able to understand what he means.

What exactly is the number 3? You may not have thought about it as being a very abstract concept. How did you first learn about three? Children learn to count before they have any notion of what the numbers mean. "One, two, three, four, five, six, seven, eight, nine, ten." What child has never felt pride in being able to count to ten? At this level counting is just a sequence of meaningless words. A parrot could be taught to do the same.

But a parrot has no concept of what those sounds mean. Nor does the little child when he or she first learns to count. But soon the child is shown a picture of three ducks, then three pigs and three elephants. She might form the idea that 'three' has something to do with animals. But when she's shown three umbrellas, three balls and three houses she begins to abstract the three-ness from these collections. But for some time she might believe that you can only count things that are the same. Perhaps the balls have different colours but all three objects are balls. It may take some time before she can count a picture of a house, a teddy bear and a horse.

Mathematics is full of analogy. Something in one branch of mathematics reminds us of something in another and that analogy can be exploited. Problems in geometry can be solved more easily by translating them into equivalent problems in algebra. At your level analogy will apply to cases where you have to solve a problem that is analogous to one whose solution you have already seen. Look out for similar patterns.

§1.5. How To Read Mathematics

One of the reasons that students have difficulty in learning maths is their inability to adjust their speed of reading to the nature of the terrain. Speed reading may be all very well for some books. Mathematics on the whole needs to be read slowly. Yet students who can read a novel at the speed of 30 pages an hour, expect to be able to do the same with mathematics.

Some non-fiction books lend themselves to be read quickly. Ideas can leap out of the page and in some cases the reader's thoughts can race ahead of the text. "Wonderful, I can see where this is leading!" But other books, and especially mathematics needs to be read at a snail's pace for much of the journey. This is especially true when the text gives way to symbols, and formulae. In fact the problem with most mathematics texts is that they're too dense, and need to be read at a speed of one page an hour! Try to read it like a novel and it's no wonder that you can't understand it.

Mathematics also needs the reader to go back frequently, to look at something that's come before. So it often needs to be "two steps forward – one step backwards". You'd never expect to complete a crossword puzzle as if you were writing a letter. You should treat mathematics as if you're doing a giant crossword.

In my notes I've tried to thin out the text with a mixture of anecdotes, and metaphors, and even illustrations that carry a bit of mild humour as they play with the words that are used. So, near to a definition of a 'supremum' (never mind what that means), I've placed an image of a mother wearing the iconic superman suit with the caption 'supermum'. These corny visual jokes serve to provide oases of space on the page where one doesn't have to think. As a result you may be able to read my notes at the speed of four pages an hour.

§1.6. Overcoming Maths Anxiety

How can you deal with maths anxiety. One way that helps is to strengthen your basic skills. Mathematics requires a combination of logic and imagination, and each of these can be strengthened even before you open the pages of a mathematics book.

Logical thinking is the ability to mindlessly follow rules. Of course a computer can do that really well so you need to be able to put yourself into 'computer mode'. A modern trend in the teaching of mathematics is to emphasise why things work. To teach a mindless recipe for solving a certain type of problem is considered to be dreadfully old-fashioned. You are supposed to *understand* the steps.

This, of course, is to misunderstand the nature of mathematics. Very few mathematical educators are real mathematicians. At the heart of mathematics is the ability to mindlessly follow certain rules. Trying to understand what you are doing when you are first shown some sort of mathematical process just makes life difficult. There is a time for teaching understanding why a certain process works but it should be at a time when the student has acquired the ability to perform the task.

The same is true of all learning. There is a lot to be said for rote learning. The problem with Victorian education is that they rarely got beyond it. Modern educational theories that emphasise creativity have added an important dimension to education but it has come at the expense of children learning rules. Asking what a certain poem means is a valuable exercise, but so is building up a small repertoire of memorised poems. Analysing why long division works is valuable but it should only come once a child has mastered the process as a mindless set of rules.

You probably know how to send an email. You mindlessly type out something like

my.friend@webcompany.com.country

Do you understand the actual process by which your email gets to your friend's computer? Do you realise that it's sent from one computer to another, maybe through countries you've never even heard of. Do you know that if your email is long it might be chopped up into several bits that are sent separately, perhaps along different routes, until they are all reassembled at the other end? No, you just type out the email address mindlessly. If you become an IT person you may find it interesting, or even important, to know what lies behind the process.

But you learn that you have to type the email address exactly, as if it was a meaningless string. If you tried to use the string

my.dear.friend@webcompany.com.country

your email would bounce back as being undeliverable.

In teaching mathematics to adults I've often encountered difficulties because they feel they should know why an algorithm works. If I try to explain the reasons it seems to confuse them in carrying out the process. They feel they can create their own rules that seem similar to mine. If I refuse to explain why, and ask them to "just do it" they feel I am treating them as children. It only works well when they allow themselves to accept the rules and acquire a technical ability to carry out the process (as a computer would) and *then*, once they can carry out the process easily, I can try to explain why it works. With adults the time between doing and understanding can be much shorter than with children.

Now here's a part of the road where there's a warning sign to slow. I remember my favourite maths lecturer, Tim Wall at Sydney University. On the blackboard he used to write a wriggly line to the left of certain pieces of mathematics to indicate "dangerous curves – slow down!"

Let me issue the same caution for the next piece. There's a strong temptation to skip over it because it looks like meaningless 'mumbo jumbo'. In fact that's what it is. It's an exercise in being able to follow rules carefully, but mindlessly. Get out a pen and paper and be prepared to spend the next ten minutes in carrying out these instructions like you would in following a recipe for a cake or instructions to assemble a piece of furniture. Write the successive changes, one under the other. If you have done it correctly you will know by the message you get on the last line.

On a sheet of paper carry out the following instructions: (Ignore the full stops at the end of each sentence.) But slow right down or you'll fly off the edge and be tempted to skip the next paragraph.

You won't understand the reason for these rules, so you are asked to follow them blindly, as if you are a computer.

Write down a string of A's and B's:

= A A A A A A B A A A B B B A B

Process it using the following rules:

(1) Delete any sequence A A A A.

(2) Delete any sequence B B.

(3) Replace any B A by A A A B.

Repeatedly use rules (1), (2) and (3), in any order
Until no further change is possible.

Let's see what happens. To make things easier to follow I will underline the sequence A A A A when rule (1) is used, B B when rule (2) is used and B A when rule (3) is used. However this underlining may be ignored.

A A A A →
B B →
B A → A A A B

Suppose we start with A A B B B A.
A A B B B A = A A B B B A Now use rule (2).
→ A A B A = A A B A Now use rule (3).
→ A A A A A B = A A A A A B Now use rule (1).
→ A B.

Suppose we begin with B A B A.
B A B A = B A B A Now use rule (3).
→ A A A B B A = A A A B B A Now use rule (2).
→ A A A A = A A A A Now use rule (1)
→ (blank)

So I see that you've skipped over all that! You realised that you'd have to slow down your reading and you'd been reading at such a speed that you found it easier to skip over all these instructions till you found out where I was going.

I included this mindless exercise to illustrate the fact that, while understanding what things mean is important in mathematics, there are times when we have to just act like an automaton, or a computer, following a series of rules accurately but mindlessly.

What does it all mean? Absolutely nothing! It was just a game with symbols, and not a very exciting game at that. Here is another task to see if you can follow rules mindlessly and accurately.

You need to be standing up with your hands by your side. There are four instructions LEFT TURN, RIGHT TURN, LOAD and UNLOAD.

LEFT TURN mean turn left 90 degrees
RIGHT TURN means turn right 90 degrees

LOAD means to hold up an arm and to extend two fingers as if you're holding a gun. UNLOAD means to 'fire' the gun, by which I mean that you drop your arm by your side.

However, whenever the gun is loaded you must do the opposite to what you're told.

If instructed to RIGHT TURN with the gun loaded you turn to the left, and vice versa. If told to LOAD when the gun is loaded you must unload. If told to UNLOAD when the gun is loaded you must load.

Remember, whenever the gun is loaded you must do the opposite to what you're told.

(1) Stand facing the door of the room with your arms by your side.
(2) LOAD
(3) LEFT TURN
(4) LOAD
(5) LEFT TURN

If you've done it correctly you should once again be standing facing the door, with your gun unloaded. Remember that at step (3) your gun is loaded so you must do a right turn. But at step (5) your gun is unloaded and so you obey the instruction and turn left.

Now repeat the instructions, but in a different order. Do (1), then (3), then (5), then (2) and then (4). This time your back will be to the door.

Does this mean anything? As far as you are concerned it is just a mindless game. It can be a fun game which can keep a group of children amused for a few minutes. However, if you were learning group theory I could explain that what lies behind this simple exercise is the group of rotations of a square.

Suppose A in our first exercise represents a RIGHT TURN and B represents a flip across the top-left to bottom-right diagonal.

Can you see why the rules now make sense. Removing A A A A corresponds to the fact that four right turns in succession is equivalent doing nothing. Similarly B B represents two 180 degree flips in succession, which again is equivalent to nothing. In the case of the children's party game B corresponds to LOAD. And under the rule that whenever the gun is loaded you have to do the opposite, LOAD followed by another LOAD, is equivalent to doing nothing.

Do you find the above explanation hard to follow? Good. It's lucky that you're not studying Group Theory. All I want you to take away from this is the realisation that it is not always necessary to understand things. The skill of accurately following a recipe is an important mathematical skill.

There is one other thing you might learn from this. When following a list of instructions, the order in which you carry out the steps is usually vitally important. BA = A A A B, which is not the same as A B. But I suppose you knew this already from your life's experience. If you rearrange the steps in a recipe you are courting disaster. Imagine cooking the cake mixture before you have mixed the ingredients!

Now following rules is only one discipline to learn. Imagination and intuition need to complement logic. You need to be able to recognise patterns and to develop a feeling for when two different things are essentially the same. This is the metaphor! Can you see any similarity in the following meaningless strings of symbols?

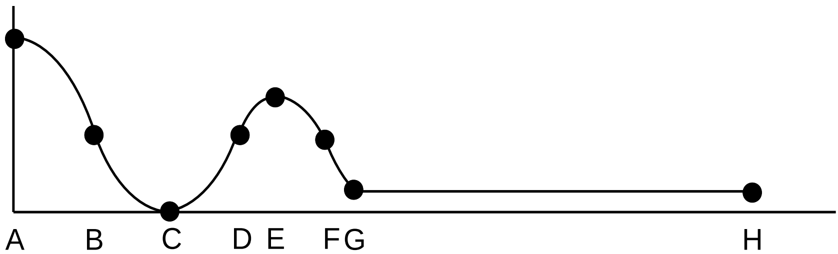
@@#$$$# and

9 9 & ! ! ! &

They're different strings, involving different symbols, but they follow the same pattern.

Here's another example to strengthen your imagination. The following graph represents the height above the water level of the feet of a bungy jumper who jumps off a bridge. The horizontal axis represents time while the vertical axis represents the height of his shoes above water level. So the first dot might be 30 metres above the water while the long horizontal line might represent a height of 3 metres above the water.

Now you must realise that the bungy jumper doesn't follow a roller coaster path. The horizontal axis represents time, not horizontal distance.



Now try to answer these questions.

(a) When does she get her feet wet?

(b) Does she go under water at any stage?

(c) Where is she falling fastest?

(d) Create a story about what might have happened, making reference to the times marked A to H.

She jumps at time A, and is dropping fastest at time B. The rubber was clearly a bit too long because her feet got wet at time C. She then bounces back up and, at time E she starts descending again. But this time she stops a few metres above the water. Probably this is because she has landed on a boat. Thinking quickly she detaches herself from the bungy elastic and remains on the deck of the boat, possibly lying flat on the deck, while the boat continues on its way.

Look for metaphors and patterns in the world around you. Perhaps you might even read some poetry in preparation for doing mathematics! Of course there's more to imagination and creativity than being able to construct or respond to metaphors. But creativity and imagination on its own is no good in mathematics. You'll never find a mathematics teacher praising a student's creativity in getting a wrong answer to an addition sum, saying "oh, that's an interesting answer". At its heart, mathematics is completely logical and unforgiving.

The difficult skill in doing mathematics is to be able to switch between mindlessly following rules and getting flashes of insight. Both are important for professional mathematicians, and both are important even for those just embarking on learning basic mathematics.

Finally, mathematics is not the dry continent that most people consider it to be. It's full of jungles and wild beasts. There is mystery in mathematics and parts of it can stretch your imagination, and your understanding of logic, till almost breaking point. Here's a taste. It's an excerpt from a book I wrote, under the pseudonym Emily Bronowski, called *Alison's Axioms*. (You can find it on this website.)

They noticed there was a book with the label 'READ ME' on the front. Inside was the title of the book: *Proof That God Exists*. It was a very thin book. In fact, apart from the title page, there were only two other pages.

On page 1 it said “God exists”. On page 2 it read “Everything in this book is false.”

“That’s a pretty silly proof,” said Ëmily.

Brother Charles was not so dismissive. “I’ve seen this book before in my seminary. Think about it logically. Page 2 is either true or it’s false.”

“Even I can unnerstand that,” said Ivy.

“Could page 2 be true?”

“No, because that would mean that it is also false,” said Alison. “It says that *everything* in the book is false and, if that were the case it *itself* would be false.”

“So page 2 must be false.”

“I suppose so, said Ëmily.

“So it is false that everything in the book is false.”

“Oh dear, that sounds like your double negative again.”

“If it’s false that everything is false then something is true.”

“Oh I get it,” said Ëmily. “One of the two pages is true, and it’s not page 2 because we said that can’t be true. So it must be page 1 that’s true. So God exists.”

“Wow,” said Alison. “I never thought you could prove that God exists by simple mathematics. But wait a minute. If I changed page 1 to ‘Santa Claus exists’ I could prove *that* true as well. Something funny’s going on here.”

Indeed something funny is indeed happening here. This is an example of places where logic has to be treated carefully. There is nothing wrong with the logic *per se*. But in logic we are not allowed to have statements that refer directly, or indirectly, to themselves.

There are great mysteries in parts of higher mathematics, including certain statements that we will never know whether or not they're true or false. Clever mathematicians have proved that it's logically impossible to prove that they're true and it's logically impossible to prove that they're false!